

ON THE COFINITENESS OF GENERALIZED LOCAL COHOMOLOGY MODULES

NGUYEN TU CUONG^A, SHIRO GOTO^B AND NGUYEN VAN HOANG^C

Abstract¹. Let R be a commutative Noetherian ring, I an ideal of R and M, N two finitely generated R -modules. The aim of this paper is to investigate the I -cofiniteness of generalized local cohomology modules $H_I^j(M, N) = \varinjlim_n \text{Ext}_R^j(M/I^n M, N)$ of M and N with respect to I . We first prove that if I is a principal ideal then $H_I^j(M, N)$ is I -cofinite for all M, N and all j . Secondly, let t be a non-negative integer such that $\dim \text{Supp}(H_I^j(M, N)) \leq 1$ for all $j < t$. Then $H_I^j(M, N)$ is I -cofinite for all $j < t$ and $\text{Hom}(R/I, H_I^t(M, N))$ is finitely generated. Finally, we show that if $\dim(M) \leq 2$ or $\dim(N) \leq 2$ then $H_I^j(M, N)$ is I -cofinite for all j .

1. INTRODUCTION

Throughout this note the ring R is commutative Noetherian. Let N be finitely generated R -modules and I an ideal of R . In [12], A. Grothendieck conjectured that if I is an ideal of R and N is a finitely generated R -module, then $\text{Hom}_R(R/I, H_I^j(N))$ is finitely generated for all $j \geq 0$. R. Hartshorne provides a counter-example to this conjecture in [13]. He also defined an R -module K to be I -cofinite if $\text{Supp}_R(K) \subseteq V(I)$ and $\text{Ext}_R^j(R/I, K)$ is finitely generated for all $j \geq 0$ and he asked the following question.

Question. *For which rings R and ideals I are the modules $H_I^j(N)$ I -cofinite for all j and all finitely generated modules N ?*

Hartshorne showed that if N is a finitely generated R -module, where R is a complete regular local ring, then $H_I^j(N)$ is I -cofinite in two cases:

- (i) I is a principal ideal (see [13, Corollary 6.3]);
- (ii) I is a prime ideal with $\dim(R/I) = 1$ (see [13, Corollary 7.7]).

Date:

¹*Key words and phrases:* Generalized local cohomology, I -cofiniteness.

2000 Subject Classification: 13C15, 13D45

^A Institute of Mathematics, 18 Hoang Quoc Viet, Hanoi, Vietnam,

E-mail : ntcuong@math.ac.vn;

^B Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan,

E-mail : goto@math.meiji.ac.jp;

^C Meiji Institute for Advanced Study of Mathematical Sciences (MIMS), Meiji University 1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan,

E-mail : nguyenvanhoang1976@yahoo.com

K. I. Kawasaki has proved that if I is a principal ideal in a commutative Noetherian ring then $H_I^j(N)$ are I -cofinite for all finitely generated R -modules N and all $j \geq 0$ (see [17, Theorem 1]). D. Delfino and T. Marley [11, Theorem 1] and K. I. Yoshida [29, Theorem 1.1] refined result (ii) to more general situation that if N is a finitely generated module over a commutative Noetherian local ring R and I is an ideal of R such that $\dim(R/I) = 1$, then $H_I^j(N)$ are I -cofinite for all $j \geq 0$. Recently, K. Bahmanpour and R. Naghipour have extended this result to the case of non-local ring; more precisely, they showed that if t is a non-negative integer such that $\dim \operatorname{Supp}(H_I^j(N)) \leq 1$ for all $j < t$ then $H_I^0(N), H_I^1(N), \dots, H_I^{t-1}(N)$ are I -cofinite and $\operatorname{Hom}(R/I, H_I^t(N))$ is finitely generated (see [2, Theorem 2.6]).

There are some generalizations of the theory of local cohomology modules. The following generalization of local cohomology theory is given by J. Herzog in [14]: Let j be a non-negative integer and M a finitely generated R -module. Then the j^{th} generalized local cohomology module of M and N with respect to I is defined by

$$H_I^j(M, N) = \varinjlim_n \operatorname{Ext}_R^j(M/I^n M, N).$$

These modules were studied further in many research papers such as: [26], [27], [3], [15], [28], [18], [16], [10], [7], [8], \dots . It is clear that $H_I^j(R, N)$ is just the ordinary local cohomology module $H_I^j(N)$.

The purpose of this paper is to investigate a similar question as above for the theory of generalized local cohomology. Our first main result is the following theorem.

Theorem 1.1. *If I is a principal ideal then $H_I^j(M, N)$ is I -cofinite for all finitely generated R -modules M, N and all j .*

As an immediate consequence of this theorem, we obtain again a theorem of K. I. Kawasaki [17, Theorem 1] (see Corollary 3.2). Moreover, Theorem 1.1 is an improvement of [6, Theorem 2.8], since we do not need the hypothesis that M has finite projective dimension as in [6]. It should be noticed that the arguments of local cohomology that used in the proof of K. I. Kawasaki [17] can not apply to proving Theorem 1.1. Because, for the case of local cohomology, if I is a principal ideal then $H_I^j(N) = 0$ for all $j > 1$. But this does not happen in the theory of generalized local cohomology, i.e. $H_I^j(M, N)$ may not vanish for $j > 1$ even if I is principal ideal. Therefore, we have to use a criterion on the cofiniteness which was invented by L. Melkersson in [23]. Here we also give a more elementary proof for this criterion (see Lemma 3.1). The next theorem is our second main result in this paper.

Theorem 1.2. *Let t be a non-negative integer such that $\dim \operatorname{Supp}(H_I^j(M, N)) \leq 1$ for all $j < t$. Then $H_I^j(M, N)$ is I -cofinite for all $j < t$ and $\operatorname{Hom}(R/I, H_I^t(M, N))$ is finitely generated.*

This theorem is an extension for generalized local cohomology modules of a result of K. Bahmanpour and R. Naghipour [2, Theorem 2.6]. In [2], they had used a basic property of local cohomology that $H_I^j(N) \cong H_I^j(N/\Gamma_I(N))$ for all $j > 0$; then it is easy to reduce to the case of $\Gamma_I(N) = 0$. But, it is not true that $H_I^j(M, N) \cong H_I^j(M, N/\Gamma_{I_M}(N))$ for

all $j > 0$ in general, where $I_M = \text{ann}_R(M/IM)$. Hence, we need to establish Lemma 2.2 which says that if t and k are non-negative integers such that $\dim \text{Supp}(H_I^j(M, N)) \leq k$ for all $j < t$ then so is $H_I^j(M, N/\Gamma_{I_M}(N))$. Moreover, in order to prove Theorem 1.2, we also need some more auxiliary lemmas such as 2.3, 2.5, 2.6 on minimax modules. Especially, by Lemma 4.2, instead of studying the cofiniteness of $H_I^j(M, N)$, we need only to prove the cofiniteness of these modules with respect to I_M . As a consequence of Theorem 1.2, we prove that if $\dim \text{Supp}(H_I^j(M, N)) \leq 1$ for all j (this is the case, for example if $\dim(N/I_M N) \leq 1$) then $H_I^j(M, N)$ is I -cofinite for all j (Corollary 4.3). This is an improvement of [6, Theorem 2.9], because our theorem does not need the hypothesis that R is complete local, M is of finite projective dimension, and I is prime ideal with $\dim(R/I) = 1$. An other consequence of Theorem 1.2 on the finiteness of Bass numbers is Corollary 4.4 which is a stronger result than the main result of S. Kawakami and K. I. Kawasaki in [18].

On the other hand, in the case of small dimension, the third author in [16, Lemma 3.1] proved that if $\dim(N) \leq 2$ then any quotient of $H_I^j(M, N)$ has only finitely many associated prime ideals for all finitely generated R -modules M and all $j \geq 0$. We can now prove a stronger result in the following theorem.

Theorem 1.3. *Assume that $\dim(M) \leq 2$ or $\dim(N) \leq 2$. Then $H_I^j(M, N)$ is I -cofinite for all j .*

As an immediate consequence of Theorem 1.3, we get a result on the cofiniteness of local cohomology modules (see Corollary 5.2). Moreover, by application of Theorems 1.2 and 1.3, we obtain a finiteness result on the set of associated prime ideals of $\text{Ext}_R^i(R/I, H_I^j(M, N))$ for all $i, j \geq 0$ when (R, \mathfrak{m}) is a Noetherian local ring and $\dim(M) \leq 3$ or $\dim(N) \leq 3$ (Corollary 5.3).

The paper is divided into five sections. In Section 2, we prove some auxiliary lemmas which will be used in the sequel. Section 3, 4 and 5 are devoted to prove three main results and its consequences.

2. AUXILIARY LEMMAS

Let R be a commutative Noetherian ring, I an ideal of R , and M, N finitely generated R -modules. We always denote by I_M the annihilator of R -module M/IM , i.e. $I_M = \text{ann}_R(M/IM)$. We first recall the following lemma.

Lemma 2.1. (cf. [9, Lemma 2.3] and [10, Lemma 2.1])

- (i) *If $I \subseteq \text{ann}(M)$ or $\Gamma_I(N) = N$ then $H_I^j(M, N) \cong \text{Ext}_R^j(M, N)$ for all $j \geq 0$.*
- (ii) *$H_I^j(M, N)$ is I_M -torsion.*

We next prove some auxiliary lemmas which will be used in sequel.

Lemma 2.2. *Let t and k be non-negative integers. If $\dim \text{Supp}(H_I^j(M, N)) \leq k$ for all $j < t$, then so is $H_I^j(M, N/\Gamma_{I_M}(N))$.*

Proof. From the short exact sequence $0 \rightarrow \Gamma_{I_M}(N) \rightarrow N \rightarrow N/\Gamma_{I_M}(N) \rightarrow 0$, we get the long exact sequence

$$\dots \rightarrow \text{Ext}_R^j(M, \Gamma_{I_M}(N)) \rightarrow H_I^j(M, N) \rightarrow H_I^j(M, \overline{N}) \rightarrow \text{Ext}_R^{j+1}(M, \Gamma_{I_M}(N)) \rightarrow \dots,$$

for all j , where $\overline{N} = N/\Gamma_{I_M}(N)$. We assume that there exists an integer $i < t$ and $\mathfrak{p} \in \text{Supp}(H_I^i(M, \overline{N}))$ such that $\dim(R/\mathfrak{p}) > k$ and $\mathfrak{p} \notin \text{Supp}(H_I^j(M, \overline{N}))$ for all $j < i$. Thus by the long exact sequence as above we obtain the following exact sequence

$$\dots \rightarrow \text{Ext}_R^j(M, \Gamma_{I_M}(N))_{\mathfrak{p}} \rightarrow H_I^j(M, N)_{\mathfrak{p}} \rightarrow H_I^j(M, \overline{N})_{\mathfrak{p}} \rightarrow \text{Ext}_R^{j+1}(M, \Gamma_{I_M}(N))_{\mathfrak{p}} \rightarrow \dots$$

Note that $H_I^j(M, N)_{\mathfrak{p}} = 0$ for all $j \leq i$, while $H_I^j(M, \overline{N})_{\mathfrak{p}} = 0$ for all $j < i$, and $H_I^i(M, \overline{N})_{\mathfrak{p}} \neq 0$. So, by the above exact sequence, we have $\text{Ext}_R^j(M, \Gamma_{I_M}(N))_{\mathfrak{p}} = 0$ for all $j \leq i$, and $\text{Ext}_R^{i+1}(M, \Gamma_{I_M}(N))_{\mathfrak{p}} \neq 0$. It implies that $\Gamma_{I_M}(N)_{\mathfrak{p}} \neq 0$ and

$$\text{depth}(\text{ann}(M)_{\mathfrak{p}}, \Gamma_{I_M}(N)_{\mathfrak{p}}) = i + 1 \geq 1.$$

Hence $\text{ann}(M)_{\mathfrak{p}} \not\subseteq \mathfrak{q}R_{\mathfrak{p}}$ for all $\mathfrak{q}R_{\mathfrak{p}} \in \text{Ass}_{R_{\mathfrak{p}}}(\Gamma_{I_M}(N)_{\mathfrak{p}})$. This contradicts with the fact that $\text{Ass}_{R_{\mathfrak{p}}}(\Gamma_{I_M}(N)_{\mathfrak{p}}) = \text{Ass}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \cap V((I_M)_{\mathfrak{p}})$ and $I_M \supseteq \text{ann}(M)$. \square

In [30], H. Zöschinger had introduced the class of minimax modules. An R -module K is said to be a *minimax module*, if there is a finitely generated submodule T of K , such that K/T is Artinian. Thus the class of minimax modules includes all finitely generated and all Artinian modules.

Lemma 2.3. *Let t be a non-negative integer such that $H_I^j(M, N)$ is I -cofinite minimax for all $j < t$. Then $\text{Hom}_R(R/I, H_I^t(M, N))$ is finitely generated. In particular, $\text{Ass}(H_I^t(M, N))$ is a finite set.*

Proof. We prove by induction on $t \geq 0$. If $t = 0$ then the result is trivial. Assume that $t > 0$ and the result holds true for $t - 1$. From the short exact sequence $0 \rightarrow \Gamma_I(N) \rightarrow N \rightarrow \overline{N} \rightarrow 0$, we get the long exact sequence

$$\text{Ext}_R^j(M, \Gamma_I(N)) \xrightarrow{f_j} H_I^j(M, N) \xrightarrow{g_j} H_I^j(M, \overline{N}) \xrightarrow{h_j} \text{Ext}_R^{j+1}(M, \Gamma_I(N)),$$

where $\overline{N} = N/\Gamma_I(N)$. For each $j \geq 0$ we split the above exact sequence into two the following exact sequences

$$0 \rightarrow \text{Im } f_j \rightarrow H_I^j(M, N) \rightarrow \text{Im } g_j \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow \text{Im } g_j \rightarrow H_I^j(M, \overline{N}) \rightarrow \text{Im } h_j \rightarrow 0.$$

Note that $\text{Im } f_j$ and $\text{Im } h_j$ is finitely generated for all $j \geq 0$. Then, $H_I^j(M, N)$ is I -cofinite if and only if $\text{Im } g_j$ is I -cofinite if and only if $H_I^j(M, \overline{N})$ is I -cofinite for all $j \geq 0$; and we get by [1, Lemma 2.1] that $H_I^j(M, N)$ is minimax if and only if so is $H_I^j(M, \overline{N})$ for all $j \geq 0$. Hence $H_I^j(M, N)$ is I -cofinite minimax if and only if so is $H_I^j(M, \overline{N})$ for all $j \geq 0$. Moreover, from the two exact sequences above, we get the following exact sequences

$$\begin{aligned} \text{Hom}(R/I, \text{Im } f_j) &\rightarrow \text{Hom}(R/I, H_I^j(M, N)) \\ &\rightarrow \text{Hom}(R/I, \text{Im } g_j) \rightarrow \text{Ext}_R^1(R/I, \text{Im } f_j) \quad \text{and} \end{aligned}$$

$$0 \rightarrow \operatorname{Hom}(R/I, \operatorname{Im} g_j) \rightarrow \operatorname{Hom}(R/I, H_I^j(M, \overline{N})) \rightarrow \operatorname{Hom}(R/I, \operatorname{Im} h_j)$$

for all $j \geq 0$. Thus we obtain that $\operatorname{Hom}(R/I, H_I^j(M, N))$ is finitely generated if and only if so is $\operatorname{Hom}(R/I, H_I^j(M, \overline{N}))$ for all $j \geq 0$. Therefore, in order to prove the lemma, we may assume that $\Gamma_I(N) = 0$. Thus, there exists $x \in I$ such that $0 \rightarrow N \xrightarrow{x} N \rightarrow N/xN \rightarrow 0$ is exact. From this, we obtain the short exact sequence

$$0 \rightarrow H_I^{t-1}(M, N)/xH_I^{t-1}(M, N) \rightarrow H_I^{t-1}(M, N/xN) \rightarrow (0 : x)_{H_I^t(M, N)} \rightarrow 0$$

Hence we get the following exact sequence

$$\begin{aligned} \operatorname{Hom}(R/I, H_I^{t-1}(M, N/xN)) &\rightarrow \operatorname{Hom}(R/I, (0 : x)_{H_I^t(M, N)}) \\ &\rightarrow \operatorname{Ext}_R^1(R/I, H_I^{t-1}(M, N)/xH_I^{t-1}(M, N)). \end{aligned}$$

By the inductive hypothesis, $\operatorname{Hom}(R/I, H_I^{t-1}(M, N/xN))$ is finitely generated. Moreover, by the assumption, $H_I^{t-1}(M, N)$ is I -cofinite minimax. It implies by [23, Corollary 4.4] that $\operatorname{Ext}_R^1(R/I, H_I^{t-1}(M, N)/xH_I^{t-1}(M, N))$ is finitely generated. Therefore, $\operatorname{Hom}(R/I, H_I^t(M, N)) \cong \operatorname{Hom}(R/I, (0 : x)_{H_I^t(M, N)})$ is finitely generated as required. \square

From Lemma 2.3, we get a short proof for the following result which was proved first by Yassemi-Khatami-Sharif in [28, Theorem 2.1].

Corollary 2.4. *If $H_I^0(M, N), H_I^1(M, N), \dots, H_I^{t-1}(M, N)$ is finitely generated then $\operatorname{Ass}(H_I^t(M, N)/T)$ is a finite set for any finitely generated submodule T of $H_I^t(M, N)$.*

Proof. By the short exact sequence $0 \rightarrow T \rightarrow H_I^t(M, N) \rightarrow H_I^t(M, N)/T \rightarrow 0$, we get the following exact sequence

$$\operatorname{Hom}(R/I, H_I^t(M, N)) \rightarrow \operatorname{Hom}(R/I, H_I^t(M, N)/T) \rightarrow \operatorname{Ext}_R^1(R/I, T).$$

Thus, since $\operatorname{Hom}(R/I, H_I^t(M, N))$ and $\operatorname{Ext}_R^1(R/I, T)$ are finitely generated by Lemma 2.3 and by the assumption of T , we obtain that $\operatorname{Hom}(R/I, H_I^t(M, N)/T)$ is also finitely generated. Thus the result follows. \square

Lemma 2.5. *Let t be a non-negative integer such that $H_I^j(M, N)$ is minimax for all $j < t$. Then $\operatorname{Hom}_R(R/I, H_I^t(M, N))$ is finitely generated and $H_I^j(M, N)$ are I -cofinite for all $j < t$.*

Proof. By Lemma 2.3, we need only to prove that $H_I^j(M, N)$ is I -cofinite for all $j < t$. We proceed by induction on j . It is clear that $H_I^0(M, N)$ is I -cofinite. Assume that $j > 0$ and the result holds true for smaller values than j . Thus we obtain that $H_I^0(M, N), \dots, H_I^{j-1}(M, N)$ are I -cofinite minimax by the inductive hypothesis and by the hypothesis. It follows by Lemma 2.3 that $\operatorname{Hom}(R/I, H_I^j(M, N))$ is finitely generated. So that $H_I^j(M, N)$ is I -cofinite by [23, Proposition 4.3] as required. \square

Lemma 2.6. *Let t be a non-negative integer such that $\operatorname{Supp}(H_I^j(M, N)) \subseteq \operatorname{Max}(R)$ for all $j < t$. Then $H_I^j(M, N)$ is Artinian for all $j < t$.*

Proof. We now prove the lemma by induction on t . If $t = 1$ then it is clear that $H_I^0(M, N)$ is Artinian. Assume that $t \geq 2$ and the lemma holds true for $t - 1$. By the inductive hypothesis, the R -modules $H_I^j(M, N)$ is Artinian for all $j < t - 1$. Therefore, by Lemma 2.5, $\text{Hom}(R/I, H_I^{t-1}(M, N))$ is finitely generated. Thus, since $\text{Supp}(\text{Hom}(R/I, H_I^{t-1}(M, N))) \subseteq \text{Max}(R)$, we obtain that $\text{Hom}(R/I, H_I^{t-1}(M, N))$ is Artinian. On the other hand, as $H_I^{t-1}(M, N)$ is I -torsion, it follows by [24, Theorem 1.3] that $H_I^{t-1}(M, N)$ is Artinian. \square

3. PROOF OF THEOREM 1.1

We first need the following lemma which has been proved in [23, Corollary 3.4] by L. Melkersson. We give here an another proof for this result with elementary arguments.

Lemma 3.1. *Let K be an R -module. Suppose $x \in I$ and $\text{Supp}(K) \subset V(I)$. If $(0 : x)_K$ and K/xK are both I -cofinite, then K must be I -cofinite.*

Proof. Let t be a non-negative integer. We need only to claim that $\text{Ext}_R^t(R/I, K)$ is finitely generated. By the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & (0 :_K x) & \rightarrow & K & \xrightarrow{x} & xK \rightarrow 0 \\ & & & & x \downarrow & \searrow x & \\ & & 0 & \longrightarrow & xK & \longrightarrow & K \rightarrow K/xK \rightarrow 0 \end{array}$$

we obtain the following commutative diagram of long exact sequences

$$\begin{array}{ccccccc} \dots & \rightarrow & \text{Ext}_R^t(R/I, (0 :_K x)) & \rightarrow & \text{Ext}_R^t(R/I, K) & \xrightarrow{x^{(t)}} & \text{Ext}_R^t(R/I, xK) \rightarrow \dots \\ & & x^{(t)} \downarrow & & & \searrow x & \\ \dots & \rightarrow & \text{Ext}_R^{t-1}(R/I, K/xK) & \rightarrow & \text{Ext}_R^t(R/I, xK) & \xrightarrow{f_t} & \text{Ext}_R^t(R/I, K) \rightarrow \dots \end{array}$$

where $x^{(t)} = \text{Ext}_R^t(R/I, x)$. Note that K/xK is I -cofinite by the hypothesis, it implies that $\text{Ext}_R^{t-1}(R/I, K/xK)$ is finitely generated. Thus $\text{Ker}(f_t)$ is finitely generated. Moreover, since the triangle is commutative, so that $x^{(t)}((0 : x)_{\text{Ext}_R^t(R/I, K)}) \subseteq \text{Ker}(f_t)$. It follows that $x^{(t)}((0 : x)_{\text{Ext}_R^t(R/I, K)})$ is finitely generated. On the other hand, $(0 : x)_K$ is I -cofinite by the hypothesis, so we obtain that $\text{Ext}_R^t(R/I, (0 : x)_K)$ is finitely generated. It implies that $\text{Ker}(x^{(t)})$ is finitely generated. Therefore, by the following exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ker}(x^{(t)}) \cap (0 : x)_{\text{Ext}_R^t(R/I, K)} &\rightarrow (0 : x)_{\text{Ext}_R^t(R/I, K)} \\ &\longrightarrow x^{(t)}((0 : x)_{\text{Ext}_R^t(R/I, K)}) \rightarrow 0, \end{aligned}$$

we obtain that $(0 : x)_{\text{Ext}_R^t(R/I, K)}$ is finitely generated. Finally, note that $x \in I$, it yields that $\text{Ext}_R^t(R/I, K) = (0 : x)_{\text{Ext}_R^t(R/I, K)}$ is finitely generated as required. \square

We now are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Assume that $I = Rx$ is a principal ideal. From the short exact sequence

$$0 \rightarrow \Gamma_I(M) \rightarrow M \rightarrow \overline{M} \rightarrow 0,$$

where $\overline{M} = M/\Gamma_I(M)$, we get by [15] the following exact sequence

$$H_I^{i-1}(\Gamma_I(M), N) \rightarrow H_I^i(\overline{M}, N) \rightarrow H_I^i(M, N) \rightarrow H_I^i(\Gamma_I(M), N)$$

for all i . Since $\Gamma_I(M) = (0 : I^k)_M$ for some positive integer k , we get by Lemma 2.1 that

$$H_I^i(\Gamma_I(M), N) = H_I^i((0 : I^k)_M, N) \cong \text{Ext}_R^i(\Gamma_I(M), N)$$

for all i . Hence $H_I^i(\Gamma_I(M), N)$ is finitely generated for all i , it follows by the above exact sequence that $H_I^i(M, N)$ is I -cofinite if and only if so is $H_I^i(\overline{M}, N)$. Hence we may assume that $\Gamma_I(M) = 0$. So that $I \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}(M)$. It implies that $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}(M)$. Thus we obtain an exact sequence

$$0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0.$$

From this we have the following exact sequence

$$0 \rightarrow H_I^{i-1}(M, N)/xH_I^{i-1}(M, N) \rightarrow H_I^i(M/xM, N) \rightarrow (0 : x)_{H_I^i(M, N)} \rightarrow 0$$

for all i . Note that, as $I = Rx$, so we obtain by Lemma 2.1 that

$$H_I^i(M/xM, N) \cong \text{Ext}_R^i(M/xM, N)$$

for all i . Hence $H_I^i(M/xM, N)$ is finitely generated for all i . Thus by the above exact sequence we obtain that

$$(0 : x)_{H_I^i(M, N)} \text{ and } H_I^i(M, N)/xH_I^i(M, N)$$

are finitely generated for all i . Therefore we get by Lemma 3.1 that $H_I^i(M, N)$ is I -cofinite for all i . \square

By replacing M by R in Theorem 1.1 we obtain a theorem of K. I. Kawasaki on the cofiniteness of local cohomology modules as follows.

Corollary 3.2. ([17, Theorem 1]) *If I is a principal ideal, then $H_I^j(N)$ is I -cofinite for all finitely generated R -module N and all j .*

4. PROOF OF THEOREM 1.2

Before proving Theorem 1.2, we need to recall some known facts on the theory of secondary representation.

In [19], I. G. Macdonald has developed the theory of attached prime ideals and secondary representation of a module, which is (in a certain sense) a dual to the theory of associated prime ideals and primary decompositions. A non-zero R -module K is called secondary if for each $a \in R$ multiplication by a on K is either surjective or nilpotent. Then $\mathfrak{p} = \sqrt{\text{ann}(K)}$ is a prime ideal and K is called \mathfrak{p} -secondary. We say that K has a secondary representation if there is a finite number of secondary submodules K_1, K_2, \dots, K_n such that $K = K_1 + K_2 + \dots + K_n$. One may assume that the prime ideals $\mathfrak{p}_i = \sqrt{\text{ann}(K_i)}$, $i = 1, 2, \dots, n$ are all distinct, and by omitting

redundant summands, that the representation is minimal. Then the set of prime ideals $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ does not depend on the representation, and it is called the set of attached prime ideals of K and denoted by $\text{Att}(K)$. Note that if A is an Artinian R -module then A has a secondary representation. The basic properties on the set $\text{Att}(A)$ of attached primes of A are referred in a paper by I. G. Macdonald [19]. If $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ is an exact sequence of Artinian R -modules then

$$\text{Att}(A_3) \subseteq \text{Att}(A_2) \subseteq \text{Att}(A_1) \cup \text{Att}(A_3).$$

Lemma 4.1. *Let x be an element of R , I an ideal of R and A an Artinian R -module. Then the following statements are true.*

- (i) *If $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Att}(A) \setminus \text{Max}(R)$, then $\ell(A/xA) < \infty$.*
- (ii) *If $(0 : I)_A$ is finitely generated, then $I \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \text{Att}(A) \setminus \text{Max}(R)$.*

Proof. (i) Assume that $\text{Att}(A) \setminus \text{Max}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Let

$$A = A_1 + \dots + A_n + B_1 + \dots + B_t$$

be a minimal secondary representation of A , where A_i is \mathfrak{p}_i -secondary and B_j is \mathfrak{m}_j -secondary for all $i = 1, \dots, n$ and all $j = 1, \dots, t$ (with $\mathfrak{m}_j \in \text{Max}(R)$ for all $j = 1, \dots, t$). Set $B = B_1 + \dots + B_t$, then $\text{Att}(B) \subseteq \text{Max}(R)$. Since $x \notin \mathfrak{p}_i$ for all $i = 1, \dots, n$, so that $xA_i = A_i$ for all $i = 1, \dots, n$. It follows that $xA = A_1 + \dots + A_n + xB$. Note that

$$\begin{aligned} A/xA &= ((A_1 + \dots + A_n + xB) + B) / (A_1 + \dots + A_n + xB) \\ &\cong B / (B \cap (A_1 + \dots + A_n + xB)). \end{aligned}$$

Therefore $\text{Att}(A/xA) \subseteq \text{Att}(B / (B \cap (A_1 + \dots + A_n + xB))) \subseteq \text{Att}(B) \subseteq \text{Max}(R)$. From this, since A/xA is Artinian, so that $\ell(A/xA) < \infty$.

(ii) We first claim that $\sqrt{\text{ann}(0 :_A I)} = \sqrt{\text{ann}(0 :_A I^n)}$ for all $n \geq 2$. Consider $n = 2$, it is clear that $\sqrt{\text{ann}(0 :_A I)} \supseteq \sqrt{\text{ann}(0 :_A I^2)}$. Conversely, for any $a \in \sqrt{\text{ann}(0 :_A I)}$ then there is an integer $t > 0$ such that $a^t(0 :_A I) = 0$. We now prove that $a^{2t}(0 :_A I^2) = 0$ (and therefore $a \in \sqrt{\text{ann}(0 :_A I^2)}$). Indeed, for any $y \in (0 :_A I^2)$, then $I^2y = 0$. So that $Iy \subseteq (0 :_A I)$, thus $a^t(Iy) = 0$. Hence $a^ty \in (0 :_A I)$, and thus $a^t(a^ty) = 0$. Therefore $a^{2t}y = 0$. We now assume that $n > 2$ and the claim is true for $n - 1$. Let $a \in \sqrt{\text{ann}(0 :_A I)}$ then by induction assumption $a \in \sqrt{(0 :_A I^{n-1})}$. Thus $a^t(0 :_A I^{n-1}) = 0$ for some $t > 0$. For any $y \in (0 :_A I^n)$, then $I^{n-1}Iy = I^n y = 0$. Hence $Iy \subseteq (0 :_A I^{n-1})$, so that $I(a^ty) = a^t(Iy) = 0$. It implies that $a^ty \in (0 :_A I)$. On the other hand, since $a \in \sqrt{\text{ann}(0 :_A I)}$, so $a^l(0 :_A I) = 0$ for some $l > 0$. Therefore $a^{t+l}y = 0$, it yields that $a \in \sqrt{(0 :_A I^n)}$. So we get the claim. Finally for any $\mathfrak{p} \in \text{Att}(A) \setminus \text{Max}(R)$ we obtain that $I \not\subseteq \mathfrak{p}$. Indeed, assume that $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \text{Att}(A) \setminus \text{Max}(R)$. Then there exists a submodule U of A such that U is \mathfrak{p} -secondary. Thus there is an integer n such that $\mathfrak{p}^n U = 0$. Hence, as $I \subseteq \mathfrak{p}$, so that $I^n U = 0$. Therefore $U = (0 :_U I^n) \subseteq (0 :_A I^n)$. Hence since $\ell(0 :_A I) < \infty$ then we get by the claim that $(0 :_A I^n)$ is of finite length. It implies that $\ell(U) < \infty$, so $\mathfrak{p} \in \text{Max}(R)$, this is a contradiction. \square

Lemma 4.2. *Let t be a non-negative integer. Then*

- (i) $H_I^t(M, N)$ is I -cofinite if and only if $H_I^t(M, N)$ is I_M -cofinite, where $I_M = \text{ann}_R(M/IM)$.
- (ii) $\text{Hom}(R/I, H_I^t(M, N))$ is finitely generated if and only if so is $\text{Hom}(R/I_M, H_I^t(M, N))$.

Proof. Set $K = H_I^t(M, N)$. Note that $\text{Supp}(K) \subseteq \text{Supp}(R/I_M) \subseteq \text{Supp}(R/I)$.

(i) If K is I -cofinite then, since $I \subseteq I_M$, we get that K is I_M -cofinite by [11, Proposition 1]. Assume that K is I_M -cofinite. Thus, as $\sqrt{I_M} = \sqrt{I + \text{ann}(M)}$, K is $(I + \text{ann}(M))$ -cofinite by [11, Proposition 1]. Let $x_1, \dots, x_t, y_1, \dots, y_s$ be generators of $I + \text{ann}(M)$ such that $I = (x_1, \dots, x_t)$ and $\text{ann}(M) = (y_1, \dots, y_s)$. Then Koszul cohomology modules $H^j(\underline{x}, y_1, \dots, y_s; K)$ are finitely generated R -modules for all j by [22, Theorem 1.1] (here we set $\underline{x} = x_1, \dots, x_t$ for short). We now claim by descending induction on l (with $0 \leq l \leq s$) that $H^j(\underline{x}, y_1, \dots, y_l; K)$ are finitely generated R -modules for all j , where we use the convention that $H^j(\underline{x}; K) = H^j(\underline{x}, y_1, \dots, y_l; K)$ if $l = 0$. If $l = s$ then the claim is clear. Suppose $l < s$ and $H^j(\underline{x}, y_1, \dots, y_{l+1}; K)$ are finitely generated R -modules for all j . We first consider the case $j = 0$. As $y_{l+1} \in \text{ann}(K)$, so we get that

$$\begin{aligned} H^0(\underline{x}, y_1, \dots, y_l; K) &\cong (0 :_K (\underline{x}, y_1, \dots, y_l)R) \\ &\cong (0 :_K (\underline{x}, y_1, \dots, y_l, y_{l+1})R) \\ &\cong H^0(\underline{x}, y_1, \dots, y_l, y_{l+1}; K). \end{aligned}$$

Thus $H^0(\underline{x}, y_1, \dots, y_l; K)$ is a finitely generated R -module. Assume that $j \geq 1$. We consider the following exact sequences (cf. [25, Section 5])

$$H^{j-1}(\underline{x}, y_1, \dots, y_l, y_{l+1}; K) \rightarrow H^j(\underline{x}, y_1, \dots, y_l; K) \xrightarrow{y_{l+1}} H^j(\underline{x}, y_1, \dots, y_l; K)$$

for all $j \geq 1$. Since $y_{l+1} \in \text{ann}(K)$, so that $y_{l+1}H^j(\underline{x}, y_1, \dots, y_l; K) = 0$. Hence, the above exact sequence implies that the following sequence

$$H^{j-1}(\underline{x}, y_1, \dots, y_l, y_{l+1}; K) \rightarrow H^j(\underline{x}, y_1, \dots, y_l; K) \rightarrow 0$$

is exact for all $j \geq 1$. From this we get by induction assumption that $H^j(\underline{x}, y_1, \dots, y_l; K)$ are finitely generated R -modules for all $j \geq 1$. Thus the claim is proved. In particular, $H^j(\underline{x}; K)$ are finitely generated R -modules for all j . Therefore, we get by [22, Theorem 1.1] again that K is I -cofinite.

(ii) We note that $\text{Hom}(R/I + \text{ann}(M), K) \cong \text{Hom}(R/I, K)$, as $\text{ann}(M) \subseteq \text{ann}(K)$. Hence, since $\sqrt{I + \text{ann}(M)} = \sqrt{I_M}$, the result follows by [11, Proposition 1]. \square

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.2, we need only to claim that $H_I^j(M, N)$ is I_M -cofinite for all $j < t$ and $\text{Hom}(R/I_M, H_I^t(M, N))$ is finitely generated, provided $\dim \text{Supp}(H_I^j(M, N)) \leq 1$ for all $j < t$ (where t is a given integer).

We prove the claim by induction on $t \geq 0$. The case of $t = 0$ is trivial. If $t = 1$ then it is clear that $H_I^0(M, N)$ is I_M -cofinite; moreover we get by Lemma 2.5 that $\text{Hom}(R/I_M, H_I^1(M, N))$ is finitely generated. Assume that $t > 1$ and the result holds

true for the case $t - 1$. From the short exact sequence $0 \rightarrow \Gamma_{I_M}(N) \rightarrow N \rightarrow \overline{N} \rightarrow 0$, we get the long exact sequence

$$\mathrm{Ext}_R^j(M, \Gamma_{I_M}(N)) \xrightarrow{f_j} H_I^j(M, N) \xrightarrow{g_j} H_I^j(M, \overline{N}) \xrightarrow{h_j} \mathrm{Ext}_R^{j+1}(M, \Gamma_{I_M}(N)),$$

where $\overline{N} = N/\Gamma_{I_M}(N)$. For each $j \geq 0$ we split the above exact sequence into two the following exact sequences

$$0 \rightarrow \mathrm{Im} f_j \rightarrow H_I^j(M, N) \rightarrow \mathrm{Im} g_j \rightarrow 0 \text{ and}$$

$$0 \rightarrow \mathrm{Im} g_j \rightarrow H_I^j(M, \overline{N}) \rightarrow \mathrm{Im} h_j \rightarrow 0.$$

Note that $\mathrm{Im} f_j$ and $\mathrm{Im} h_j$ is finitely generated for all $j \geq 0$. Then, for each $j < t$, we obtain that $H_I^j(M, N)$ is I_M -cofinite if and only if so is $H_I^j(M, \overline{N})$. On the other hand, we get by Lemma 2.2 that $\dim \mathrm{Supp}(H_I^j(M, \overline{N})) \leq 1$ for all $j < t$. Therefore, in order to prove the theorem for the case of $t > 1$, we may assume that $\Gamma_{I_M}(N) = 0$. Hence $I_M \not\subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(N)} \mathfrak{p}$. Set

$$X = \bigcup_{j=0}^{t-1} \mathrm{Supp}(H_I^j(M, N)) \text{ and } S = \{\mathfrak{p} \in X \mid \dim(R/\mathfrak{p}) = 1\}.$$

Thus $S \subseteq \bigcup_{j=0}^{t-1} \mathrm{Ass}(H_I^j(M, N))$. Note that $H_I^j(M, N)$ is I_M -cofinite for all $j < t - 1$ and $\mathrm{Hom}(R/I_M, H_I^{t-1}(M, N))$ is finitely generated by the inductive hypothesis. It implies that $\bigcup_{j=0}^{t-1} \mathrm{Ass}(H_I^j(M, N))$ is a finite set, and so S is a finite set. Assume that $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$. Then it is clear that

$$\mathrm{Supp}_{R_{\mathfrak{p}_k}}(H_{IR_{\mathfrak{p}_k}}}^j(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k})) \subseteq \mathrm{Max}(R_{\mathfrak{p}_k})$$

for all $j < t$ and all $k = 1, \dots, n$. From this, we get by Lemma 2.6 that $H_{IR_{\mathfrak{p}_k}}}^j(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k})$ is Artinian for all $j < t$ and all $k = 1, \dots, n$. Note that $V(I_M) \subseteq V(I)$. Hence, it implies by Lemma 2.5 and [11, Proposition 1] that $\mathrm{Hom}(R_{\mathfrak{p}_k}/(I_M)R_{\mathfrak{p}_k}, H_{IR_{\mathfrak{p}_k}}}^j(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k}))$ is finitely generated for all $j < t$ and all $k = 1, \dots, n$. Therefore it yields by Lemma 4.1(ii) that

$$V((I_M)R_{\mathfrak{p}_k}) \cap \mathrm{Att}_{R_{\mathfrak{p}_k}}(H_{IR_{\mathfrak{p}_k}}}^j(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k})) \subseteq \mathrm{Max}(R_{\mathfrak{p}_k})$$

for all $j < t$ and all $k = 1, \dots, n$. Let

$$T = \bigcup_{j=0}^{t-1} \bigcup_{k=1}^n \{\mathfrak{q} \in \mathrm{Spec} R \mid \mathfrak{q}R_{\mathfrak{p}_k} \in \mathrm{Att}_{R_{\mathfrak{p}_k}}(H_{IR_{\mathfrak{p}_k}}}^j(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k}))\}.$$

Then we have $T \cap V(I_M) \subseteq S$. We now choose an element $x \in I_M$ such that

$$x \notin \left(\bigcup_{\mathfrak{p} \in T \setminus V(I_M)} \mathfrak{p} \right) \cup \left(\bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(N)} \mathfrak{p} \right).$$

Thus, we have the short exact sequence $0 \rightarrow N \xrightarrow{x} N \rightarrow N/xN \rightarrow 0$. It implies the following exact sequence

$$H_I^j(M, N) \xrightarrow{x} H_I^j(M, N) \rightarrow H_I^j(M, N/xN) \rightarrow H_I^{j+1}(M, N)$$

for all $j \geq 0$. Thus, we have an exact sequence

$$(1) \quad 0 \rightarrow H_I^j(M, N)/xH_I^j(M, N) \xrightarrow{\alpha_j} H_I^j(M, N/xN) \xrightarrow{\beta_j} (0 : x)_{H_I^{j+1}(M, N)} \rightarrow 0$$

for all $j \geq 0$. Note that $\dim \text{Supp}(H_I^j(M, N/xN)) \leq 1$ for all $j < t - 1$ by the above exact sequence and by the hypothesis. So that, we get by the induction assumption that $H_I^0(M, N/xN), H_I^1(M, N/xN), \dots, H_I^{t-2}(M, N/xN)$ are I_M -cofinite and $\text{Hom}(R/I_M, H_I^{t-1}(M, N/xN))$ is finitely generated. Moreover, also by the induction assumption, we have $H_I^0(M, N), H_I^1(M, N), \dots, H_I^{t-2}(M, N)$ are I_M -cofinite and $\text{Hom}(R/I_M, H_I^{t-1}(M, N))$ is finitely generated. For each $j < t$, we set $L_j = H_I^j(M, N)/xH_I^j(M, N)$. By the choice of x and by Lemma 4.1, we obtain that $(L_j)_{\mathfrak{p}_k}$ has finite length for all $j < t$ and all $k = 1, \dots, n$. From this by the Noetherianness of $(L_j)_{\mathfrak{p}_k}$, there exists a finitely generated submodule L_{jk} of L_j such that $(L_j)_{\mathfrak{p}_k} = (L_{jk})_{\mathfrak{p}_k}$ for any $j < t$ and any $k = 1, \dots, n$. Let $L'_j = L_{j1} + L_{j2} + \dots + L_{jn}$. Then L'_j is a finitely generated submodule of L_j satisfying the following inclusion

$$\text{Supp}(L_j/L'_j) \subseteq X \setminus \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\} \subseteq \text{Max}(R)$$

for all $j < t$. For each $j < t$, we set $N_j = H_I^j(M, N/xN)$ and $N'_j = \alpha_j(L'_j)$. Then N'_j is a finitely generated submodule of N_j and the following sequence

$$(2) \quad 0 \rightarrow L_j/L'_j \xrightarrow{\alpha_j^*} N_j/N'_j \xrightarrow{\beta_j^*} (0 : x)_{H_I^{j+1}(M, N)} \rightarrow 0$$

is exact. We now prove that L_j is minimax for all $j < t$. Look at the exact sequence

$$\text{Hom}(R/I_M, N_j) \rightarrow \text{Hom}(R/I_M, N_j/N'_j) \rightarrow \text{Ext}_R^1(R/I_M, N'_j).$$

For any $j < t$, since N'_j is finitely generated and $\text{Hom}(R/I_M, N_j)$ is finitely generated, so that $\text{Hom}(R/I_M, N_j/N'_j)$ is finitely generated. Hence we obtain by the sequence (2) that $\text{Hom}(R/I_M, L_j/L'_j)$ is finitely generated for all $j < t$. While $\text{Supp}(L_j/L'_j) \subseteq \text{Max}(R)$ and L_j/L'_j is I_M -torsion, so that L_j/L'_j is Artinian by [24, Theorem 1.3] for all $j < t$. Thus L_j is minimax for all $j < t$. Consider again the exact sequence (1), that is the following sequence

$$(1') \quad 0 \rightarrow L_j \xrightarrow{\alpha_j} N_j \xrightarrow{\beta_j} (0 : x)_{H_I^{j+1}(M, N)} \rightarrow 0.$$

As $\text{Hom}(R/I_M, N_j)$ is finitely generated for all $j < t$, then so is $\text{Hom}(R/I_M, L_j)$ for all $j < t$. From this, we obtain by [23, Proposition 4.3] that L_j is I_M -cofinite for all $j < t$. Keep in mind that N_j is I_M -cofinite for all $j < t - 1$. Thus, from the sequence (1'), we have that $(0 : x)_{H_I^j(M, N)}$ is I_M -cofinite for all $j < t$. In particular, $(0 : x)_{H_I^{t-1}(M, N)}$ and $H_I^{t-1}(M, N)/xH_I^{t-1}(M, N) = L_{t-1}$ are I_M -cofinite. It implies that $H_I^{t-1}(M, N)$ is I_M -cofinite by Lemma 3.1. Thus $H_I^j(M, N)$ is I_M -cofinite for all $j < t$. On the other hand, by the sequence (1') when $j = t - 1$, we have the following exact sequence

$$\text{Hom}(R/I_M, N_{t-1}) \rightarrow \text{Hom}(R/I_M, (0 : x)_{H_I^t(M, N)}) \rightarrow \text{Ext}_R^1(R/I_M, L_{t-1}).$$

Thus, since $\text{Hom}(R/I_M, N_{t-1})$ is finitely generated and L_{t-1} is I_M -cofinite, so it yields that $\text{Hom}(R/I_M, H_I^t(M, N)) = \text{Hom}(R/I_M, (0 : x)_{H_I^t(M, N)})$ is finitely generated. Hence the claim is proved, and the proof of Theorem 1.2 is complete. \square

In [6, Theorem 2.9], Divaani Aazar and Sazeeleh showed that if \mathfrak{p} is a prime ideal in a complete local ring (R, \mathfrak{m}) with $\dim(R/\mathfrak{p}) = 1$, then $H_{\mathfrak{p}}^j(M, N)$ is \mathfrak{p} -cofinite for all $j \geq 0$ whenever M has finite projective dimension. Here, even if without the hypothesis completeness of the ring and the finiteness of projective dimension of M , we still obtain the following result.

Corollary 4.3. *If $\dim \operatorname{Supp}(H_I^j(M, N)) \leq 1$ for all j (this is the case, for example if $\dim(N/I_M N) \leq 1$), then $H_I^j(M, N)$ is I -cofinite for all $j \geq 0$.*

We now recall the notion of Bass number: let K be an R -module, i an integer and \mathfrak{p} a prime ideal, then the i -th Bass number $\mu^i(\mathfrak{p}, K)$ of K with respect to \mathfrak{p} was defined by $\mu^i(\mathfrak{p}, K) = \dim_{k(\mathfrak{p})}(\operatorname{Ext}_R^i(R/\mathfrak{p}, K)_{\mathfrak{p}})$. In [18], S. Kawakami and K. I. Kawasaki proved that if M has finite projective dimension and $\dim(R/I) = 1$ then $\mu^i(\mathfrak{p}, H_I^j(M, N))$ is finite for all $i, j \geq 0$ and all $\mathfrak{p} \in \operatorname{Spec}(R)$. The next corollary is a generalization of this result.

Corollary 4.4. *Assume that $\dim \operatorname{Supp}(H_I^j(M, N)) \leq 1$ for all j (this is the case, for example if $\dim(N/I_M N) \leq 1$). Then $\mu^i(\mathfrak{p}, H_I^j(M, N))$ is finite for all $i, j \geq 0$ and all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Proof. If $I \not\subseteq \mathfrak{p}$ then $\mu^i(\mathfrak{p}, H_I^j(M, N)) = 0$. If $I \subseteq \mathfrak{p}$ then $\operatorname{Supp}(R/\mathfrak{p}) \subseteq \operatorname{Supp}(R/I)$, so that $\operatorname{Ext}_R^i(R/\mathfrak{p}, H_I^j(M, N))$ is finitely generated for all i, j by Corollary 4.3 and [11, Proposition 1]. Therefore $\mu^i(\mathfrak{p}, H_I^j(M, N))$ is finite for all i, j , as required. \square

5. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. We first consider the case of $\dim(M) \leq 2$. By the short exact sequence $0 \rightarrow \Gamma_I(M) \rightarrow M \rightarrow \overline{M} \rightarrow 0$ where $\overline{M} = M/\Gamma_I(M)$, we get the following exact sequence

$$H_I^{j-1}(\Gamma_I(M), N) \xrightarrow{f_j} H_I^j(\overline{M}, N) \xrightarrow{g_j} H_I^j(M, N) \xrightarrow{h_j} H_I^j(\Gamma_I(M), N)$$

(following [15]). It implies the following exact sequences

$$0 \rightarrow \operatorname{Im} f_j \rightarrow H_I^j(\overline{M}, N) \rightarrow \operatorname{Im} g_j \rightarrow 0 \text{ and}$$

$$0 \rightarrow \operatorname{Im} g_j \rightarrow H_I^j(M, N) \rightarrow \operatorname{Im} h_j \rightarrow 0.$$

Since $\Gamma_I(M) = (0 : I^k)_M$ for some integer k , so that

$$H_I^j(\Gamma_I(M), N) = H_{I^k}^j((0 : I^k)_M, N) = \operatorname{Ext}_R^j(\Gamma_I(M), N)$$

for all j by Lemma 2.1. Thus $\operatorname{Im} f_j$ and $\operatorname{Im} h_j$ are finitely generated for all j . So by the above exact sequences we obtain that $H_I^j(\overline{M}, N)$ is I -cofinite if and only if so is $H_I^j(M, N)$. Therefore we may assume that $\Gamma_I(M) = 0$. Then there exists $x \in I$ such that x is an M -regular element. From the short exact sequence $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ we get the following exact sequence

$$H_I^j(M/xM, N) \longrightarrow (0 : x)_{H_I^j(M, N)} \longrightarrow 0.$$

Since $\dim(M/xM) \leq 1$, so that $\dim \text{Supp}((0 : x)_{H_I^j(M, N)}) \leq 1$. Note that $H_I^j(M, N)$ is I -torsion and $x \in I$. Thus

$$\dim \text{Supp}(H_I^j(M, N)) = \dim \text{Supp}((0 : x)_{H_I^j(M, N)}) \leq 1$$

for all j . From this we obtain by Corollary 4.3 that $H_I^j(M, N)$ is I -cofinite for all j . For the rest of this proof, we consider the case of $\dim(N) \leq 2$. By the short exact sequence $0 \rightarrow \Gamma_I(N) \rightarrow N \rightarrow \overline{N} \rightarrow 0$ where $\overline{N} = N/\Gamma_I(N)$, we get the following exact sequence

$$\text{Ext}_R^j(M, \Gamma_I(N)) \xrightarrow{u_j} H_I^j(M, N) \xrightarrow{v_j} H_I^j(M, \overline{N}) \xrightarrow{w_j} \text{Ext}_R^{j+1}(M, \Gamma_I(N)).$$

It implies the following exact sequences

$$0 \rightarrow \text{Im } u_j \rightarrow H_I^j(M, N) \rightarrow \text{Im } v_j \rightarrow 0 \text{ and}$$

$$0 \rightarrow \text{Im } v_j \rightarrow H_I^j(M, \overline{N}) \rightarrow \text{Im } w_j \rightarrow 0.$$

Thus $\text{Im } u_j$ and $\text{Im } w_j$ are finitely generated for all j . So by the above exact sequences we obtain that $H_I^j(M, \overline{N})$ is I -cofinite if and only if so is $H_I^j(M, N)$. Hence we may assume that $\Gamma_I(N) = 0$. So we can take $y \in I$ such that y is an N -regular element. From the exact sequence $0 \rightarrow N \xrightarrow{y} N \rightarrow N/yN \rightarrow 0$ we have an exact sequence as follow

$$H_I^j(M, N/yN) \longrightarrow (0 : y)_{H_I^{j+1}(M, N)} \longrightarrow 0$$

for all j . So that $\dim \text{Supp}(H_I^j(M, N)) = \dim \text{Supp}((0 : y)_{H_I^{j+1}(M, N)}) \leq 1$ for all $j \geq 1$. Note that $H_I^0(M, N) = \text{Hom}(M, \Gamma_I(N)) = \text{Hom}(M, 0) = 0$. Thus $\dim \text{Supp}(H_I^j(M, N)) \leq 1$ for all j . From this we get by Corollary 4.3 that $H_I^j(M, N)$ is I -cofinite for all j , and this finishes the proof of Theorem 1.3. \square

As immediate consequences of Theorem 1.3 we obtain the following results.

Corollary 5.1. *If $\dim(R) \leq 2$ the $H_I^j(M, N)$ is I -cofinite for all j , and all finitely generated R -modules M, N .*

Corollary 5.2. *If $\dim(N) \leq 2$ the $H_I^j(N)$ is I -cofinite for all j .*

We next consider furthermore a consequence of Theorem 1.2 and 1.3 on the finiteness of associated primes of generalized local cohomology modules. We first recall that an R -module K is called *weakly Laskerian* if any quotient module of K has finitely many associated primes (cf. [11]). Note that, all Artinian modules, all finitely generated modules, and all modules with finite support are weakly Laskerian. Moreover, if $0 \rightarrow K_1 \rightarrow K_2 \rightarrow K_3 \rightarrow 0$ is an exact sequence, then K_2 is weakly Laskerian if and only if K_1 and K_3 are both weakly Laskerian. Note that if R is a Noetherian local ring and $\dim(N) \leq 3$ then the third author proved in [16, Theorem 1.1] that the modules $H_I^j(M, N)$ has only finitely many associated prime ideals for all j . In the following, we obtain a stronger result.

Corollary 5.3. *Assume that (R, \mathfrak{m}) is a Noetherian local ring. If $\dim(M) \leq 3$ or $\dim(N) \leq 3$ then $\text{Ext}_R^i(R/I, H_I^j(M, N))$ is weakly Laskerian for all $i, j \geq 0$. In particular, $\text{Ass}_R(H_I^j(M, N))$ is a finite set for all $j \geq 0$.*

Proof. Assume that $\dim(M) \leq 3$. By similar arguments as in the proof of Theorem 1.3, we obtain the following exact sequences

$$0 \rightarrow \text{Im } f_j \rightarrow H_I^j(\overline{M}, N) \rightarrow \text{Im } g_j \rightarrow 0 \text{ and}$$

$$0 \rightarrow \text{Im } g_j \rightarrow H_I^j(M, N) \rightarrow \text{Im } h_j \rightarrow 0,$$

where $\overline{M} = M/\Gamma_I(M)$. Thus we get the following exact sequences

$$\dots \rightarrow \text{Ext}_R^i(R/I, \text{Im } f_j) \rightarrow \text{Ext}_R^i(R/I, H_I^j(\overline{M}, N)) \rightarrow \text{Ext}_R^i(R/I, \text{Im } g_j) \rightarrow \dots \text{ and}$$

$$\dots \rightarrow \text{Ext}_R^i(R/I, \text{Im } g_j) \rightarrow \text{Ext}_R^i(R/I, H_I^j(M, N)) \rightarrow \text{Ext}_R^i(R/I, \text{Im } h_j) \rightarrow \dots$$

Moreover, note that $\text{Im } f_j$ and $\text{Im } h_j$ are finitely generated for all j . It follows that $\text{Ext}_R^i(R/I, H_I^j(M, N))$ is weakly Laskerian if and only if so is the module $\text{Ext}_R^i(R/I, H_I^j(\overline{M}, N))$. Therefore we may assume that $\Gamma_I(M) = 0$. Thus we get an exact sequence $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ where $x \in I$ is a regular element of M . It implies that $H_I^j(M/xM, N) \rightarrow (0 : x)_{H_I^j(M, N)} \rightarrow 0$ is an exact sequence. Hence, as $\dim(M/xM) \leq 2$, so we obtain that

$$\dim \text{Supp}(H_I^j(M, N)) \leq 2 \text{ for all } j \geq 0.$$

For the case $\dim(N) \leq 3$, by similar arguments as above we may reduce to the hypothesis that $\Gamma_I(N) = 0$. Then by the following exact sequence

$$H_I^j(M, N/yN) \rightarrow (0 : y)_{H_I^{j+1}(M, N)} \rightarrow 0$$

with $y \in I$ is an N -regular element, we get that $\dim \text{Supp}(H_I^j(M, N)) \leq 2$ for all $j \geq 0$. Therefore, for the rest of this proof, we need only to claim the weakly Laskerianity of $\text{Ext}_R^u(R/I, H_I^v(M, N))$ for all $u, v \geq 0$ provided that

$$\dim \text{Supp}(H_I^j(M, N)) \leq 2 \text{ for all } j \geq 0.$$

Note that $H_I^j(M, N) \otimes_R \widehat{R} \cong H_I^j(\widehat{M}, \widehat{N})$. Therefore, in view of [20, Lemma 2.1], we can assume that R is complete with \mathfrak{m} -adic topology. We now claim the weakly Laskerianity of $\text{Ext}_R^u(R/I, H_I^v(M, N))$ by way of contradiction. For any integers u, v , we set $K = \text{Ext}_R^u(R/I, H_I^v(M, N))$. Assume that there exists a submodule T of K such that $\text{Ass}(K/T)$ is an infinite set. Then there is a countably infinite subset $\{\mathfrak{p}_l\}_{l \in \mathbb{N}}$ of $\text{Ass}(K/T)$ such that $\mathfrak{p}_l \neq \mathfrak{m}$ for all $l \in \mathbb{N}$. Let $S = R \setminus \bigcup_{l \in \mathbb{N}} \mathfrak{p}_l$. Then S is a multiplicative closed subset of R . Since $\{\mathfrak{p}_l\}_{l \in \mathbb{N}} \subseteq \text{Ass}(K/T)$, so that $\{S^{-1}\mathfrak{p}_l\}_{l \in \mathbb{N}} \subseteq \text{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}T)$. Thus $\text{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}T)$ is an infinite set. On the other hand, as $\mathfrak{m} \not\subseteq \mathfrak{p}_l$ for all $l \in \mathbb{N}$, we get by [21, Lemma 3.2] that $\mathfrak{m} \not\subseteq \bigcup_{l \in \mathbb{N}} \mathfrak{p}_l$; and so that $\mathfrak{m} \cap S \neq \emptyset$. It implies that $\dim \text{Supp}(H_{S^{-1}I}^j(S^{-1}M, S^{-1}N)) \leq 1$ for all $j \geq 0$. From this, we obtain by Corollary 4.3 that

$$S^{-1}K = \text{Ext}_{S^{-1}R}^u(S^{-1}R/S^{-1}I, H_{S^{-1}I}^v(S^{-1}M, S^{-1}N))$$

is finitely generated. It implies that $S^{-1}K/S^{-1}T$ is finitely generated. Hence $\text{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}T)$ is a finite set. On the other hand, by the hypothesis of T , the set $\text{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}T)$ is infinite. Hence we obtain a contradiction, and the claim follows. The last conclusion is clear. \square

REFERENCES

- [1] K. Bahmanpour and R. Naghipour, *On the cofiniteness of local cohomology modules*, Proc. Amer. Math. Soc., (7) **136** (2008), 2359-2363.
- [2] K. Bahmanpour and R. Naghipour, *Cofiniteness of local cohomology modules for ideals of small dimension*, J. Algebra, **321** (2009), 1997-2011.
- [3] M. H. Bijan-Zadeh, *A common generalization of local cohomology theories*, Glasgow Math. J., **21** (1980), 173-181.
- [4] M. Brodmann and A. L. Faghani, *A Finiteness result for associated primes of local cohomology modules*, Proc. Amer. Math. Soc., (10) **128** (2000), 2851-2853.
- [5] M. Brodmann and R.Y. Sharp, "Local cohomology: an algebraic introduction with geometric applications," *Cambridge University Press*, (1998).
- [6] K. Divaani Aazar and R. Sazeedeh, *Cofiniteness of generalized local cohomology modules*, Coll. Algebra, **99** (2004), 283-290.
- [7] M. Chardin and K. Divaani Aazar, *Generalized local cohomology and regularity of ext modules*, J. Algebra, (11) **319** (2008), 4780-4797.
- [8] M. Chardin and K. Divaani Aazar, *A duality theorem for generalized local cohomology*, Proc. Amer. Math. Soc., **136** (2008), 2749-2754.
- [9] N. T. Cuong and N. V. Hoang, *Some finite properties of generalized local cohomology modules*, East-West J. Math. (2) **7** (2005), 107-115.
- [10] N. T. Cuong and N. V. Hoang, *On the vanishing and the finiteness of supports of generalized local cohomology modules*, Manuscripta Math., (1) **126** (2008), 59-72 .
- [11] D. Delfino and T. Marley, *Cofinite modules and local cohomology*, J. Pure Appl. Algebra **121** (1997), 45-52.
- [12] A. Grothendieck, *Cohomologie local des faisceaux coherents et théorèmes de Lefschetz locaux et globaux (SGA2)*, North-Holland, Amsterdam, 1968.
- [13] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math., **9** (1970), 145-164.
- [14] J. Herzog, *Komplexe, Auflösungen und Dualität in der Lokalen Algebra*, Habilitationsschrift, Universität Regensburg, 1970.
- [15] J. Herzog and N. Zamani, "Duality and vanishing of generalized local cohomology", *Arch. Math. J.*, (5) **81** (2003), pp. 512-519.
- [16] N. V. Hoang, *On the associated primes and the supports of generalized local cohomology modules*, Acta Math. Vietnam., **33** (2008), 163-171.
- [17] K. I. Kawasaki, *Cofiniteness of local cohomology modules for principal ideals*, Bull. London. Math. Soc. **30** (1998), 241-246.
- [18] S. Kawakami and K. I. Kawasaki, *On the finiteness of Bass numbers of generalized local cohomology modules*, Toyama Math. J., **29** (2006), 59-64.
- [19] I. G. Macdonald, *Secondary representation of modules over a commutative ring*, Symp. Math. XI (1973), 23-43.
- [20] T. Marley, *Associated primes of local cohomology module over rings of small dimension*, Manuscripta Math., (4) **104** (2001), 519-525.
- [21] T. Marley and J.C. Vassilev, *Cofiniteness and associated primes of local cohomology modules*, J. Algebra **256** (2002), 180-193.
- [22] L. Melkersson, *Properties of cofinite modules and applications to local cohomology*, Math. Proc. Camb. Phil. Soc. **125** (1999), 417-423.
- [23] L. Melkersson, *Modules cofinite with respect to an ideal*, J. Algebra **285** (2005), 649-668.

- [24] L. Melkersson, *On asymptotic stability for sets of prime ideals connected with the powers of an ideal*, Math. Proc. Cambridge Philos. Soc., **107** (1990), 267-271.
- [25] L. Melkersson, *Some applications of a criterion for artinianness of a module*, J. Pure Appl. Alg., **101** (1995), 291-303.
- [26] N. Suzuki, *On the generalized local cohomology and its duality*, J. Math. Kyoto Univ., **18** (1978), 71-78.
- [27] S. Yassemi, *Generalized section functors*, J. Pure Appl. Alg., **95** (1994), 103-119.
- [28] S. Yassemi, L. Khatami and T. Sharif, *Associated primes of generalized local cohomology modules*, Comm. Algebra, (1) **30** (2002), 327-330.
- [29] K. I. Yoshida, *Cofiniteness of local cohomology modules for ideals of dimension one*, Nagoya Math. J., **147** (1997) 179-191.
- [30] H. Zöschinger, *Minimax modules*, J. Algebra. **102** (1986), 1-32.